# A lattice non-perturbative definition of an $SO(10)$ chiral gauge theory and its induced standard model

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The standard model is a chiral gauge theory where the gauge fields couple to the right-hand and the left-hand fermions differently. The standard model is defined perturbatively and describes all elementary particles (except gravitons) very well. However, for a long time, we do not know if we can have a non-perturbative definition of standard model as a Hamiltonian quantum mechanical theory. In this paper, we propose a way to give a modified standard model (with 48 two-component Weyl fermions) a non-perturbative definition by embedding the modified standard model into a $SO(10)$ chiral gauge theory. We show that the $SO(10)$ chiral gauge theory can be put on a lattice (a 3D spatial lattice with a continuous time) if we allow fermions to interact. Such a non-perturbatively defined standard model is a Hamiltonian quantum theory with a finite-dimensional Hilbert space for a finite space volume. More generally, using the defining connection between gauge anomalies and the symmetry-protected topological orders, one can show that any truly anomaly-free chiral gauge theory can be non-perturbatively defined by putting it on a lattice in the same dimension.

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**Introduction**: The $U(1) \times SU(2) \times SU(3)$ standard model[1–6] is the theory which is believed to describe all elementary particles (except gravitons) in nature. However, the standard model was defined only perturbatively initially, via the perturbative expansion of the gauge coupling constant. Even though the perturbative expansion is known to diverge, if we only keep the first a few orders of the perturbative expansion, the standard model produces results that compare very well with experiments.

So the "perturbatively defined standard model" (keeping only first a few orders of the perturbation) is a theory of nature. However, the "perturbatively defined standard model" is certainly not a "Hamiltonian quantum theory" (by keeping only a few orders of the perturbation, the probability may not even be conserved). "Hamiltonian quantum theory" is a quantum theory with
(1) a finite dimensional Hilbert space for all the states,
(2) a local Hamiltonian operator for the time evolution,
(3) operators to describe the physical quantities.
So far, we do not know if there is a non-perturbatively defined standard model which is a Hamiltonian quantum theory. In this paper, we like to address this issue. We will propose a way to obtain a non-perturbative definition of the standard model that defines the standard model as a Hamiltonian quantum theory.

Defining standard model non-perturbatively is a well-known long standing problem, which is referred generally as chiral-fermion/chiral-gauge problem. There are many previous researches that try to solve this general problem. There are lattice gauge theory approaches,[7] which fail since they cannot reproduce chiral couplings between the gauge field and the fermions. There are domain-wall fermion approaches.[8,9] But the gauge fields in the domain-wall fermion approaches propagate in one-higher dimension: 4+1D. There are also overlap-fermion approaches.[10–15] However, the path-integral in overlap-fermion approaches may not describe a Hamiltonian quantum theory (for example, the total Hilbert space in the overlap-fermion approaches, if exist, may not have a finite dimension, even for a space-lattice of a finite size). There are also the mirror fermion approach used in Ref. 16–19, which start with a lattice model containing chiral fermions coupled to gauge theory *and* the chiral conjugated mirror sector. Then, one tries to include proper direct interaction or boson mediated interactions[20,21] between fermions hopping to gap out the mirror sector only without breaking the gauge symmetry. However, later work either fail to demonstrate[22–24] or argue that it is almost impossible to gap out the mirror sector without breaking the gauge symmetry in some mirror fermion models.[25] Some of those negative results are based on some particular choices of fermion interactions for some particular chiral gauge theories.

In Ref. 26, a deeper understanding of gauge anomalies and gravitational anomalies is obtained through symmetry-protected topological (SPT) orders and topological orders in one-higher dimensions. This leads to a particular way to construct mirror fermion models and a particular way to construct interactions between fermions. Such a construction leads to a complete solution of chiral-fermion/chiral-gauge problem:

> By definition, any chiral gauge theory can be non-perturbatively defined as a low energy effective theory of a lattice theory of finite degrees of freedom per site by including proper interactions between fermions, provided that the chiral gauge theory is free of *all* anomalies.

In other words, the lattice gauge theory approach actually works (*i.e.* can be used to define any truly-anomaly-free chiral-gauge theories), provided that we include proper interactions between fermions. In Ref. 27, we show that a 1+1D $U(1)$ chiral fermion/boson theory is free of *all* the $U(1)$ gauge anomalies if it is free of the Adler-Bell-Jackiw (ABJ) $U(1)$ gauge anomaly.[28,29] Thus, all the ABJ-anomaly-free 1+1D $U(1)$ chiral fermion/boson theories can be non-perturbatively defined by $U(1)$ lattice theories.

However, in general, we do not know how to check if a chiral gauge theory is free of *all* anomalies.[30] So the above result is hard to use. To address this problem, in this paper, we provide some support for the following conjecture:

**Conjecture:** A chiral fermion theory in $d$-dimensional space-time with a gauge group $G$ is free of all gauge and gravitational anomalies if (1) there exist (possibly symmetry breaking) mass terms that make all the fermions massive, and (2) $\pi_n(G/G_{\text{grnd}}) = 0$ for $n \leq d+1$, where $G_{\text{grnd}}$ is the unbroken symmetry group.

Such a conjecture allows us to show that

the $SO(10)$ chiral fermion theory in the $SO(10)$ grand unification[31] can appear as a low energy effective theory of a lattice gauge model in 3D space with a continuous time, which has a finite number of degrees of freedom per site.

In other words, the mirror fermion approach works for the $SO(10)$ chiral fermion theory. Following Ref. 26, we propose a way to design a proper fermion interactions that can gap out the mirror sector only, without breaking the $SO(10)$ gauge symmetry.

By embedding the modified standard model into the $SO(10)$ grand unification model,[31,32] the above non-perturbatively defined $SO(10)$ chiral fermion theory gives a non-perturbative definition of a modified standard model. Compare to the standard model, the modified standard model contains a total of 48 two-component Weyl fermions (one extra neutrino for each family).

In the rest of this paper, we will first give a brief review of the connection between gauge anomalies and SPT orders.[26] Next we will describe a particular construction that gives a general non-perturbative definition of all weak-coupling chiral gauge theories that are free of all anomalies. Then, as a key result, we will show that, using such a construction, the modified standard model (with 48 two-component Weyl fermions) and its corresponding $SO(10)$ chiral gauge theory can be defined as a 3D lattice $SO(10)$ gauge model with a continuous time (*i.e.* the low energy effective theory of the lattice $SO(10)$ gauge model is the modified standard model).

**Gauge anomalies and SPT orders in one-higher dimension**: To understand gauge anomalies in weak-coupling gauge theories, we can take the zero coupling limit. In this limit, the gauge theory become a theory with a global symmetry described by group $G$. Through such a limit, we find that we can gain a systematic understanding of gauge anomalies through SPT states.[26]

What are SPT states? SPT states[33,34] are short-range entangled states[35] with an on-site symmetry[36–38] described by a symmetry group $G$. It was shown that different SPT states in $(d+1)$-dimensional space-time are classified by group cohomology class $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$.[36–38] The SPT states have very special low energy boundary effective theories, where the symmetry $G$ in the bulk is realized as a *non-on-site* symmetry on the boundary. (We will also refer non-on-site symmetry as anomalous symmetry.) It turns out that the non-on-site symmetry (or the anomalous symmetry) on the boundary is not "gaugable". If we try to gauge the non-on-site symmetry, we will get an anomalous gauge theory, as demonstrated in Ref. 37,39–42 for $G = U(1), SU(2)$. This relation between SPT states and gauge anomalies on the boundary of the SPT states allows us to obtain a systematic understanding of gauge anomalies via the SPT states in one-higher dimension. In particular, one can use different elements in group cohomology class $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$ to classify (at least partially) different bosonic gauge anomalies for gauge group $G$ in $d$-dimensional space-time. This result applies for both continuous and discrete gauge groups. The free part of $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$, Free[$\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$], classifies the well known ABJ anomalies[28,29] for both bosonic and fermionic systems. The torsion part of $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$ correspond to new types of gauge anomalies beyond the Adler-Bell-Jackiw anomalies (which will be called non-ABJ gauge anomalies).[43]

Note that the global symmetry of the bulk SPT state in the $d+1$ dimensional space-time is on-site and gaugable. If we gauge the global symmetry, we obtain a non-perturbative definition of an anomalous gauge theory in $d$ dimensional space-time. The $d$ dimensional anomalous gauge theory is defined as the boundary theory of the $d+1$ dimensional (gauged) SPT state. We see that an anomalous gauge theory is not well defined in the same dimension, but it can be defined as the boundary theory of a (gauged) SPT state in one-higher dimension. In the next section, we will show that an anomaly-free chiral gauge theory can always be defined as a lattice gauge theory in the same dimension.

**A non-perturbative definition of anomaly-free chiral gauge theories**: Motivated by the connection between the chiral gauge theories in $d$-dimensional space-time and the SPT states in $(d+1)$-dimensional space-time, we like to show that one can give a non-perturbative definition of any anomaly-free chiral gauge theories.

Let us start with a SPT state in $(d+1)$-dimensional space-time with an on-site symmetry $G$ (see Fig. 1a). We assume that the SPT state is described by a cocycle $\nu \in \mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$. On the $d$-dimensional boundary, the low energy effective theory will have a non-on-site symmetry (*i.e.* an anomalous symmetry) $G$. Here we will assume that the $d$-dimensional boundary excitations are gapless. After "gauging" the on-site symmetry $G$ in the $(d+1)$-dimensional bulk, we get a chiral gauge theory on the $d$-dimensional boundary whose anomaly is described by the cocycle $\nu$.

Then let us consider a stacking of a few bosonic SPT states in $(d+1)$-dimensional space-time described by cocycles $\nu_i \in \mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$ where the interaction between the SPT states are weak (see Fig. 1b). We also assume

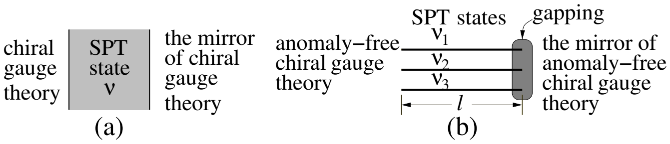


FIG. 1: (a) A SPT state described by a cocycle $\nu \in \mathcal{H}^{d+1}(G,\mathbb{R}/\mathbb{Z})$ in $(d+1)$-dimensional space-time. After "gauging" the on-site symmetry $G$, we get a bosonic chiral gauge theory on one boundary and the "mirror" of the bosonic chiral gauge theory on the other boundary. (b) A stacking of a few SPT states in $(d+1)$-dimensional space-time described by cocycles $\nu_i$. If $\sum_i \nu_i = 0$, then after "gauging" the on-site symmetry $G$, we get a anomaly-free chiral gauge theory on one boundary. We also get the "mirror" of the anomaly-free chiral gauge theory on the other boundary, which can be gapped without breaking the "gauge symmetry".

that $\sum_i \nu_i = 0$. In this case, if we turn on a proper $G$-*symmetric* interaction on one boundary, we can fully gap the boundary excitations in such a way that the ground state is not degenerate. (Such a gapping process also do not break the $G$ symmetry.) Thus the gapping process does not leave behind any low energy degrees of freedom on the gapped boundary. Now we "gauge" the on-site symmetry $G$ in the $(d+1)$-dimensional bulk. The resulting system is a non-perturbative definition of anomaly-free chiral gauge theory described by $\nu_i$ with $\sum \nu_i = 0$. Since the thickness $l$ of the $(d+1)$-dimensional bulk is finite (although $l$ can be large so that the two boundaries are nearly decoupled), the system actually has a $d$-dimensional space-time. In particular, due to the finite $l$, the gapless gauge bosons of the gauge group $G$ only live on the $d$-dimensional boundary.

The same approach also works for fermionic systems. We can start with a few fermionic SPT states in $(d+1)$-dimensional space-time described by super-cocycles $\nu_i$[44] that satisfy $\sum \nu_i = 0$ (*i.e.* the combined fermion system is free of all the gauge anomalies). If we turn on a proper $G$-symmetric interaction on one boundary, we can fully gap the boundary excitations in such a way that the ground state is not degenerate and does break the symmetry $G$. In this case, if we gauge the bulk on-site symmetry, we will get a non-perturbative definition of anomaly-free fermionic chiral gauge theory.

**A non-perturbative definition of an $SO(10)$ chiral gauge theory**: To define an $SO(10)$ chiral gauge theory[31] in 4-dimensional space-time, we start with a free fermion hopping model on a 4-dimensional space lattice (with a continuous time). We design the free fermion hopping model such that there is a fermion band gap in the bulk and there is a single two-component gapless Weyl fermion mode on the boundary (see appendix A for a particular construction).[45,46] We also assume that the 4-dimensional space lattice form a slab of thickness $l$. The massless Weyl fermions on one boundary is described by by the following Hamiltonian

$$H = -\psi^\dagger \,\mathrm{i}\,\sigma^i \partial_i \psi \tag{1}$$

where $\psi$ is a two-component Weyl fermion operator, and $\sigma^l$, $l = 1, 2, 3$ are the Pauli matrices. We will call $\psi$ the right-hand Weyl fermions. The massless Weyl fermions on the other boundary is described by left-hand Weyl fermions with a Hamiltonian

$$H = -\tilde{\psi}^\dagger \,\mathrm{i}\,(\sigma^i)^* \partial_i \tilde{\psi}. \tag{2}$$

Next, we take 16 copies of the above theory, which will lead to 16 gapless right-hand Weyl fermions on one boundary

$$H = -\psi_\alpha^\dagger \,\mathrm{i}\,\sigma^i \partial_i \psi_\alpha, \quad \alpha = 1, \cdots, 16. \tag{3}$$

and 16 gapless left-hand Weyl fermions on the other boundary. Such 16 fermions will form the 16-dimensional spinor representation of $SO(10)$. We note that, by construction, the free fermion hopping model on the 4-dimensional space lattice has the $SO(10)$ symmetry, which is an on-site symmetry.

Then, we add an $SO(10)$ symmetric interaction between the left-hand Weyl fermions described by (2) on one boundary. If the interaction can fully gap out the left-hand Weyl fermions (*i.e.* give all the left-hand Weyl fermions a finite mass) without breaking the $SO(10)$ symmetry, then, the only low energy excitations are the massless right-hand Weyl fermions that form the spinor representation of $SO(10)$. Since $l$ is finite, we can view the 4-dimensional slab as a 3-dimensional lattice. Thus, we obtain a lattice model of interacting fermions in 3-dimensional space, such that the low energy excitations of the model are the right-hand Weyl fermions forming the spinor representation of $SO(10)$. The lattice model also has the $SO(10)$ on-site symmetry. After gauging the $SO(10)$ on-site symmetry in 4+1D lattice theory, we obtain a non-perturbative definition of $SO(10)$ chiral gauge theory in terms of a lattice gauge theory in 3-dimensional space.

The key step in the above construction is to add a proper interaction between the left-hand Weyl fermions on one boundary to gap out all the left-hand Weyl fermions without breaking the $SO(10)$ symmetry. Is this possible? If the $SO(10)$ chiral fermion theory (with right-hand Weyl fermion in 16-dimensional representation of $SO(10)$) is free of all the gauge anomalies, then almost by definition, there will exist a proper interaction between the Weyl fermions on one boundary to gap out all the Weyl fermions without breaking the $SO(10)$ symmetry. We know that the $SO(10)$ chiral fermion theory is free of all ABJ gauge anomalies and free of all gravitational anomalies (since the chiral fermion can all be gapped if we break the $SO(10)$ symmetry). However, we do not know if the $SO(10)$ chiral fermion theory is free of all potential nonABJ anomalies (such as global gauge anomalies). In the following, we will propose a way to design the interaction between the Weyl fermions so that the interaction can gap out all the Weyl fermions on one boundary without breaking the $SO(10)$ symmetry. This suggests that the $SO(10)$ chiral fermion theory is free of all gauge anomalies.

One way to obtain such an interaction is to introduce real scaler fields $\phi^a$, $a = 1, \cdots, 10$, in the 10-dimensional representation of $SO(10)$ and construct the following interacting theory

$$H = -\tilde{\psi}^\dagger_\alpha \mathrm{i}(\sigma^i)^* \partial_i \tilde{\psi}_\alpha + H(\phi^a) + \tilde{\psi}^T \epsilon C \gamma_a \phi^a \tilde{\psi} + h.c. \quad (4)$$

where $\epsilon = \mathrm{i}\sigma^2$ acting on the Weyl spinor index. Here $H(\phi^a)$ is the Hamiltonian for the scaler fields $\phi^a$, and the 16-by-16 matrices $C$ and $\gamma_a$ are chosen such that $\tilde{\psi}^T \epsilon C \phi^a \tilde{\psi}$ form the 10-dimensional representation of $SO(10)$ (see appendix B for details).[47] $C\gamma_a\phi^a$ can be viewed as a hermitian matrix with eight eigenvalues equal to $\sqrt{\phi^a\phi^a}$ and eight eigenvalues equal to $-\sqrt{\phi^a\phi^a}$. Therefore, the term $\tilde{\psi}^T \epsilon C \gamma_a \phi^a \tilde{\psi} + h.c.$ generate a mass $M = \sqrt{\phi^a\phi^a}$ for all the 16 Weyl fermions if the $\phi^a$ field is a non-zero constant. The non-zero constant $\phi^a$ field break the $SO(10)$ symmetry. The fact that the 16 Weyl fermions can be fully gapped implies that they are free of gravitational anomalies.

The Hamiltonian $H(\phi^a)$ for the real scaler field is chosen to make $\phi^a\phi^a = M^2 \neq 0$ without breaking the $SO(10)$ symmetry $\langle\phi^a\rangle = 0$. So the orientation of the $\phi^a$ field can fluctuate freely within a sphere $S_9$ in 10-dimensional space. We also assume that the correlation length $\xi$ of the $\phi^a$ field is much larger than the lattice constant. In this case, we expect the term $\tilde{\psi}^T \epsilon C \gamma_a \phi^a \tilde{\psi} + h.c.$ generate a mass $M \sim \sqrt{\phi^a\phi^a}$ for all the 16 Weyl fermions even when the $\phi^a$ field is fluctuating and $\langle\phi^a\rangle = 0$.

However, the above argument may fail if the fluctuating $\phi^a$ field in 4-dimensional space-time contains defects where $\phi^a = 0$. Those defects with $\phi^a = 0$ can give rise to massless (or gapless) fermionic excitations. Point-defect in space-time with $\phi^a = 0$ (such as instantons) can exist if $\pi_3(S_9) \neq 0$, line-defect in space-time with $\phi^a = 0$ (such as "hedgehog" solitons) can exist if $\pi_2(S_9) \neq 0$, membrane-defect in space-time with $\phi^a = 0$ (such as vortex lines) can exist if $\pi_1(S_9) \neq 0$, 3D-brane-defect in space-time with $\phi^a = 0$ (such as domain walls) can exist if $\pi_0(S_9) \neq 0$. However, $\pi_d(S_9) = 0$ for $0 \leq d < 9$. So there are no defects with $\phi^a = 0$. We may assume the fluctuating $\phi^a$ field satisfying $\phi^a \neq 0$ anywhere in space-time.

The above argument may also fail if the effective Lagrangian for the non-vanishing fluctuating $\phi^a$ field in 4-dimensional space-time contains a Wess-Zumino-Witten (WZW) term (the WZW term can be well-defined for non-vanishing $\phi^a$ field),[48,49] after we integrating out the massive fermions in the 4+1D bulk. In this case, $\phi^a$ field may not have a gapped phase that do not break the symmetry, as discussed in Ref. 36–38,49. However, since $\pi_5(S_9) = 0$, the non-vanishing $\phi^a$ field in 4-dimensional space-time cannot have any WZW term.

The above considerations make us to believe that the term $\tilde{\psi}^T \epsilon C \gamma_a \phi^a \tilde{\psi} + h.c.$ does generate a mass $M \sim \sqrt{\phi^a\phi^a}$ for all the 16 Weyl fermions even when $\langle\phi^a\rangle = 0$ and the $SO(10)$ symmetry is not broken. The fact that the 16 Weyl fermions can be fully gapped without breaking the $SO(10)$ symmetry implies that they are free of *all* $SO(10)$ gauge anomalies.[26]

The above argument can be generalized to other symmetries, which leads to the conjecture stated at the begining of the paper. In the above $SO(10)$ example, the symmetry breaking fields $\phi^a$ can generate the (Higgs) mass terms in the conjecture that give all the fermions a mass gap. The unbroken symmetry group $G_{\text{grnd}}$ in the conjecture is $SO(9)$. The configurations of the symmetry breaking fields generated by the $SO(10)$ rotations forms a space $G/G_{\text{grnd}} = SO(10)/SO(9) = S_9$.

We will try to apply our anomaly-free conditions to some other chiral fermion theories. If the conditions are satisfied, then the chiral fermion theory is free of all anomalies. If not, the theory may or may not have anomalies. For a chiral fermion theory with $U(1)$ gauge symmetry, any mass term will break the $U(1)$ symmetry, and thus $G_{\text{grnd}} = 1$ (*i.e.* trivial). We have $\pi_1(G/G_{\text{grnd}}) = \mathbb{Z}$, and the condition (2) is not satisfied. So the theory can be anomalous which is a correct result. Next, let us consider a chiral fermion theory with a $SU(2)$ gauge symmetry. The theory contains two right-hand fermions forming an $SU(2)$ doublet. The theory also contains two left-hand fermions which are $SU(2)$ singlet. We can make all the fermions massive by breaking the $SU(2)$ symmetry completely (*i.e.* $G_{\text{grnd}} = 1$). Since $\pi_3(G/G_{\text{grnd}}) = \pi_3[SU(2)] = \mathbb{Z}$, the the condition (2) is not satisfied for 2-dimensional space-time and above. So the theory can be anomalous in 2-dimensional space-time and above, which is again correct. The above two examples demonstrate that our argument does not apply for known anomalous theories.

**Summary**: In this paper, we proposed a way to construct a lattice gauge model to non-perturbatively define a 3+1D $SO(10)$ chiral gauge theory with two-component massless Weyl fermions in the 16-dimensional spinor representation of $SO(10)$. The close connection between gauge anomalies and the SPT orders allows us to show that any chiral gauge theory can be non-perturbatively defined by putting it on a lattice of the same dimension, as long as the chiral gauge theory is free of all anomalies. Such construction is achieved by adding a proper strong interaction among the fermions. As a key result, we propose a general way to add/design such an interaction.

The 3+1D $SO(10)$ chiral gauge theory on lattice can be combined with Higgs fields to break the $SO(10)$ gauge "symmetry" to $U(1)\times SU(2)\times SU(3)$ gauge "symmetry", which leads to the modified standard model and its non-perturbative definition on lattice. Such a procedure was studied under the $SO(10)$ grand unified theory.[31]

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## Appendix A: The lattice model

The lattice model in 4D space, whose boundary gives rise to a single massless Weyl fermion, has the following form

$$H = H_{\text{hop}} + H_{\text{int}}, \tag{A1}$$

where

$$H_{\text{hop}} = \sum_{ij} (t_{ij} c^\dagger_{\alpha,i} c_{\alpha,j} + h.c.) \tag{A2}$$

is a lattice fermion hopping model with 16 fermion orbitals (labled by $\alpha = 1, \cdots, 16$) per site. $H_{\text{int}}$ describe the interaction between the fermions.

Let us first construct

$$H^1_{\text{hop}} = \sum_{ij} (t_{ij} c^\dagger_i c_j + h.c.) \tag{A3}$$

which has one fermion orbital per site. To construct $H^1_{\text{hop}}$, let us introduce

$$\begin{aligned} &\Gamma^1 = \sigma^1 \otimes \sigma^3, \quad \Gamma^2 = \sigma^2 \otimes \sigma^3, \\ &\Gamma^3 = \sigma^0 \otimes \sigma^1, \quad \Gamma^4 = \sigma^0 \otimes \sigma^2, \quad \Gamma^5 = \sigma^3 \otimes \sigma^3, \end{aligned} \tag{A4}$$

which satisfy

$$\{\Gamma^i, \Gamma^j\} = 2\delta_{ij}. \tag{A5}$$

In the $k$ space, the lattice model $H^1_{\text{hop}}$ is given by the following one-body Hamiltonian

$$\begin{aligned} &H(k_1,k_2,k_3,k_4) \\ &= 2[\Gamma^1\sin(k_1)+\Gamma^2\sin(k_2)+\Gamma^3\sin(k_3)+\Gamma^4\sin(k_4)] \\ &\quad + 2\Gamma^5[\cos(k_1)+\cos(k_2)+\cos(k_3)+\cos(k_4)-3]. \end{aligned} \tag{A6}$$

Since the band structure of such a 4D hopping model (A6) is designed to have a non-trivial twist, the 4D lattice model will have one two-component massless Weyl fermion on its 3-dimensional surface, appearing at the zero energy (single-body energy).[45,46]

Let us consider a 4-dimensional lattice formed by stacking two 3-dimensional cubic lattices. We then put the above 4-dimensional lattice fermion hopping model on such a 4-dimensional lattice which has only two layers in the $x^4$-direction. The one-body Hamiltonian in the $(k_1,k_2,k_3)$-space is given by the following 8-by-8 matrix

$$H(k_1,k_2,k_3) = \begin{pmatrix} M_1 & M_2 \\ M_2^\dagger & M_1 \end{pmatrix} \tag{A7}$$

where

$$\begin{aligned} M_1 &= 2[\Gamma^1\sin(k_1)+\Gamma^2\sin(k_2)+\Gamma^3\sin(k_3)] \\ &\qquad + 2\Gamma^5[\cos(k_1)+\cos(k_2)+\cos(k_3)-3], \\ M_2 &= -\mathrm{i}\,\Gamma^4+\Gamma^5. \end{aligned} \tag{A8}$$

We find that the above fermion hopping model give rise to one two-component massless Weyl fermion on each of the two surfaces of the 4D lattice. (A surface is a 3D cubic lattice.) The Weyl fermion on one boundary is left-hand Weyl fermion and the Weyl fermion on the other boundary is right-hand Weyl fermion.

The above hopping model is defined on a 4D lattice with only two layers of 3D cubic lattices. We may also construct a hopping model on a 4D lattice with $l$ layers. In this case, we still get one two-component Weyl fermion on each of the two surfaces of the 4D lattice. However, the two-component Weyl fermions on different surfaces has a mixing of order $\mathrm{e}^{-l}$, which gives the fermion a Dirac mass of order $\mathrm{e}^{-l}$. (Our two-layer model is fine tuned to make such a mixing vanishes.)

Then we put 16 copies of the above hopping model $H^1_{\text{hop}}$ together to obtain a hopping model $H_{\text{hop}}$ with an $SO(10)$ symmetry (where fermions form the 16-dimensional spinor representation of $SO(10)$). Next we try to include a proper $SO(10)$ symmetric interaction among fermions on only one boundary to give those, say left-hand, fermion a mass term of order cut-off scale without breaking the $SO(10)$ symmetry. In the main text, we discussed how to design such an interaction [via scalar fields $\phi^a$ in the 10-dimensional representation of $SO(10)$]. Since the target space of the scalar fields $\phi^a$ is $S_9$ which has trivial homopoty group $\pi_d(S_9)=0$ for $d<9$, we argue that such a scalar field can generate an interaction term $H_{\text{int}}$ which gives the left-hand fermion on one boundary a mass term of order cut-off scale without breaking the $SO(10)$ symmetry.

Since the mixing of the fermions on the two boundaries is of order $\mathrm{e}^{-l}$, the interaction on one boundary will only induce a weak $SO(10)$ symmetric interaction of order $\mathrm{e}^{-l}$ on the other boundary. Since all the interactions are irrelavent, any weak interactions cannot give the right-hand fermions on the other boundary a mass term. The right-hand fermions on the other boundary will be massless.

Once we put the right-hand Weyl fermions on lattice with the full $SO(10)$ symmetry (realized as an on-site symmetry), then it is easy to gauge the global (on-site) $SO(10)$ symmetry to obtain a lattice $SO(10)$ gauge model which produces right-hand massless Weyl fermions coupled to $SO(10)$ gauge field, at low energies.

## Appendix B: $SO(10)$ spinor representations

To understand the $SO(10)$ spinor representations,[47] let us introduce $\gamma$-matrices $\gamma_a$, $a=1,\cdots,10$:

$$\begin{aligned} \gamma_{2k-1} &= \underbrace{\sigma^0\otimes\cdots\otimes\sigma^0}_{k-1\ \sigma^0\text{'s}}\otimes\sigma^1\otimes\underbrace{\sigma^3\otimes\cdots\otimes\sigma^3}_{5-k\ \sigma^3\text{'s}} \\ \gamma_{2k} &= \underbrace{\sigma^0\otimes\cdots\otimes\sigma^0}_{k-1\ \sigma^0\text{'s}}\otimes\sigma^2\otimes\underbrace{\sigma^3\otimes\cdots\otimes\sigma^3}_{5-k\ \sigma^3\text{'s}} \\ k &= 1,\cdots,5, \end{aligned} \tag{B1}$$

which satisfy

$$\{\gamma_a,\gamma_b\} = 2\delta_{ab}, \quad \gamma_a^\dagger = \gamma_a. \tag{B2}$$

Here $\sigma^0$ is the 2-by-2 identity matrix and $\sigma^l$, $l=1,2,3$ are the Pauli matrices. The 45 hermitian matrices

$$\Gamma_{ab} = \frac{\mathrm{i}}{2}[\gamma_a,\gamma_b] = \mathrm{i}\,\gamma_a\gamma_b, \quad a<b, \tag{B3}$$

generate a 32-dimensional representation of $SO(10)$: $\mathrm{e}^{\mathrm{i}\,\theta^{ab}\Gamma_{ab}}$, $\theta_{ab}=-\theta_{ba}$. The above 32-dimensional representation is reducible. To obtain irreducible representation, we introduce

$$\begin{aligned} \gamma_{\text{FIVE}} &= (-)^5\gamma_1\otimes\cdots\otimes\gamma_{10} = \underbrace{\sigma^3\otimes\cdots\otimes\sigma^3}_{5\ \sigma^3\text{'s}}, \\ (\gamma_{\text{FIVE}})^2 &= 1, \quad \mathrm{Tr}\gamma_{\text{FIVE}} = 0. \end{aligned} \tag{B4}$$

We see that $\{\gamma_{\text{FIVE}},\gamma_a\} = [\gamma_{\text{FIVE}},\Gamma_{ab}] = 0$. This allows us to obtain two 16-dimensional irreducible representations

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\,\theta^{ab}\Gamma^+_{ab}}:\ \Gamma^+_{ab} &= \frac{1+\gamma_{\text{FIVE}}}{2}\Gamma_{ab}\frac{1+\gamma_{\text{FIVE}}}{2}, \\ \mathrm{e}^{\mathrm{i}\,\theta^{ab}\Gamma^-_{ab}}:\ \Gamma^-_{ab} &= \frac{1-\gamma_{\text{FIVE}}}{2}\Gamma_{ab}\frac{1-\gamma_{\text{FIVE}}}{2}. \end{aligned} \tag{B5}$$

The two 16-dimensional irreducible representations are related. Let us introduce

$$C = \sigma^2\otimes\sigma^1\otimes\sigma^2\otimes\sigma^1\otimes\sigma^2, \tag{B6}$$

which satisfies

$$C^{-1}\Gamma^*_{ab}C = -\Gamma_{ab}, \qquad C^{-1}\gamma^*_a C = -\gamma_a,$$
$$C^{-1}\gamma_{\text{FIVE}}C = -\gamma_{\text{FIVE}}. \tag{B7}$$

If the Weyl fermion operators $\psi_+$ form the 16-dimensional irreducible representation $\Gamma^+_{ab}$, then $\psi_- = C\psi^*_+$ is the other 16-dimensional irreducible representation $\Gamma^-_{ab}$.

Using the above results, we can show that $\psi^T_+ \epsilon C\gamma_a \psi_+$ form a 10-dimensional representation of $SO(10)$, since

$$[\Gamma_{ab}, \gamma_c] = -2\mathrm{i}(\delta_{ac}\gamma_b - \delta_{bc}\gamma_a). \tag{B8}$$

The above leads to

$$\begin{aligned}
&\psi^T_+ \epsilon C\gamma_a \psi_+ = \psi^T_+ \epsilon \frac{1+\gamma_{\text{FIVE}}}{2} C\gamma_a \frac{1+\gamma_{\text{FIVE}}}{2}\psi_+ \\
&\to \psi^T_+ \epsilon\, \mathrm{e}^{\mathrm{i}\,\theta^{ab}\Gamma^T_{ab}} \frac{1+\gamma_{\text{FIVE}}}{2} C\gamma_a \frac{1+\gamma_{\text{FIVE}}}{2}\, \mathrm{e}^{\mathrm{i}\,\theta^{ab}\Gamma_{ab}}\psi_+ \\
&= \psi^T_+ \epsilon \frac{1+\gamma_{\text{FIVE}}}{2} CC^{-1}\, \mathrm{e}^{\mathrm{i}\,\theta^{ab}\Gamma^*_{ab}} C\gamma_a\, \mathrm{e}^{\mathrm{i}\,\theta^{ab}\Gamma_{ab}} \frac{1+\gamma_{\text{FIVE}}}{2}\psi_+ \\
&= \psi^T_+ \epsilon \frac{1+\gamma_{\text{FIVE}}}{2} C\, \mathrm{e}^{-\mathrm{i}\,\theta^{ab}\Gamma_{ab}}\gamma_a\, \mathrm{e}^{\mathrm{i}\,\theta^{ab}\Gamma_{ab}} \frac{1+\gamma_{\text{FIVE}}}{2}\psi_+ \\
&= G^b_a(\theta_{ab})\psi^T_+ \epsilon \frac{1+\gamma_{\text{FIVE}}}{2} C\gamma_b \frac{1+\gamma_{\text{FIVE}}}{2}\psi_+, \\
&= G^b_a(\theta_{ab})\psi^T_+ \epsilon C\gamma_b \psi_+,
\end{aligned} \tag{B9}$$

where the 10-by-10 matrix $G(\theta_{ab}) \in SO(10)$. Here, we may view $C\gamma_b$ and $\Gamma_{ab}$ as 16-by-16 matrices acting within the 16-dimensional space with $\frac{1+\gamma_{\text{FIVE}}}{2} = 1$. Note that $C\gamma_b$ and $\Gamma_{ab}$ commute with $\frac{1+\gamma_{\text{FIVE}}}{2}$. When viewed as such a 16-by-16 matrix, $C\gamma_{10}$ is a real symmetric matrix with eight eigenvalues equal to 1 and eight eigenvalues equal to $-1$.